

ARTS2818

5i ghfu`]Ub'Dc`]hjWg

HYfa 'C bY''##'&\$%

7ci fgY⁻Cj Yfj]Yk

'GhJZZ'7cbhJWh'8YhJ]`g'

'7cbj Ybcfg'

B Ua Y	9a U]`	5j U]`UV]`]hm	@cWUhjcb	D\cbY
< Y`Yb'Df]b[`Y	\"df]b[`Y4 i bgk "YXi "Ui	6mUddc]bha Ybh	A 6 '%()	-',)&'(*

GWXcc`7cbHJWhi±bZcfa Uhjcb

GWkcc``cZGcWJU``GWJYbWYg

Fcca '%) -

A cfj Yb 6fck b 7 &\$

Ya U]`. 'gcgg4 i bgk "YXi "Ui

d\cbY. '\$&'-',) '% \$+

7ci fgY 8YW] g

7 fYX]hDc]bhg**

Gi a a UfmcZh\Y 7ci fgY

Gi VYWh5fYU. Ba (f[Ue S`V; fWd Sf[a`S^DWSf[a`e

H\]g`Wci fgY`]bhfcXi Wyg`mci `hc`h\Y`]bgh]hi h]cbgždfUW]Wyg`UbX`df]bW]d`Yg`cZfYdfYgYbhUh]j Y`XYa cWfUWmUbX ^ gh]WYžk]h\]b`U`Wca dUfUh]j Y`WcbhYl h'`Mci `k]```YUfb`UVci hh\Y`5 i ghfU`]Ub`Wcbgh]hi h]cbU``ZfUa Yk cf_žh\Y gYdUfUh]cb`cZdck Yfg`UbX`h\Y`^ X]W]U``gmghYa žh\Y`´″i bk f]hhYb`Wcbgh]hi h]cb´´´`cZ[cj Yfb]b[`Wcbj Ybh]cbgžUbX\i a Ub`f][\hg"`Mci `k]``Wca Y`fc`UddfYW]Uh%6i ghfU`]Ub`Y`YWcfU``gmghYa g`UbX`VY\Uj]ci fžUg`k Y```Ug`h\Y

5ggYgga Ybh

Ghi XYbhgʻUfYʻYl dYWYX'hcʻgi Va]hik f]hhYbʻk cf_ʻZcfʻh\YʻWti fgYʻ]bʻY`YWffcb]WZcfa žʻj]UʻHi fb]h]bʻ]bʻA ccX`Y" DʻYUgYʻgi Va]hUgʻUʻk cfXʻZ]ʻYʻfh\Uhi]gžʻ"XcWcfʻ"XcW žʻB C Hʻ"dXZŁ"ʻG@E L bdah[VWe egbbadʻ XadefgVW fe [ˈge[ˈY Fgdˈ;f;`ł Sf\hdg.#ghi XYbh'i bgk "YXi "Ui #i fb]h]b!gi ddcfhž FZ[e i WhbSYW[`UʻgVWe [ˈXad_Sf[a` a` Zai fa YW WbSfWS` V [ˈfWbbdWfad[Y[ˈSf[k dWbadfe [ˈFgdˈ;f;`ž Egbbadʻ [e S'ea ShS[′ST'WXba_fZWUW foS^Fgdˈ;f;` : Wb 5 W fWdSf\hd.#hi fb]h]b"Wta #bSi g#gi ddcfh*\Y`d!\WbhYf"

5 'gi [[YghYX'hYa d`UhY'Zcf'h\Y'YggUmž'UbX'U'a Uf_]b['fi Vf]Wžk]```VY'gi dd`]YX'Vmh\Y'7 ci fgY'7 ccfX]bUhcf"' \= \] g'h\Y'fYgdcbg]V]`]hmcZghi XYbhg'hc'_YYd'U'WcdmcZh\Y]f'k cf_']b'WUgY'cZ`cgg"'Gi Va]hhYX'k cf_'k]```VY a Uf_YX'UbX'fYhi fbYX'Y'YWfcb]WU``mžk \]W\`YbUV`Yg'ZYYXVUW_hc'VY'a i W\'a cfY'h\cfci [\'fUbX``Y[]V`YŁ"'\= Z mci '\Uj Y'UbmX]ZZ]W\`'h]Yg'k]h\'Y'YWfcb]Wgi Va]gg]cbž'd`YUgY'gYY'h\Y'7 ci fgY'7 ccfX]bUhcf'Ug'gccb'Ug dcgg]V`Y"

 $\label{thm:cfx} $$HY'Zc``ck]b['HUV'Y'gYhg'ci hk cfX``]a]hg'UbX'Xi Y'XUhYg'Zcf'h\Y'j Uf]ci g'dUfhg'cZUggYgga Ybh']b'h\Y'Wci fgYž Uh'U'[`UbWY'' $$$

.

5ggYgga Ybh HJg_	@Yb[h\	K Y][\h]b[@YUfb]b[CiHWcaYg 5ggYggYX	8i Y'XUHY
Cb`]bY'ei]n	CbY'\cif	&\$1	% (K YY_'(
Cb`]bY'ei]n	CbY'\cif	&\$1	% (K YY_'%\$
A UĉfYggUm	&) \$\$ k cfXg) \$1	% (%A Um
Hi hcf]U`dUfj\dUfj\dUfj\dUfj\cb`Ž fYdcfh	♥ Ydcfh') \$\$ k cfXgQ	% 1	%(%A Um

 $+ \frac{1}{2} \frac$

5ggYgga YbhHUg_g

5ggYgga Ybh'hUg_	K Y][\h	8iY'8UhY	Gh XYbh@YUfb]b[CihWcaYg5ggYggYX
HYghg [*]	(\$1	% "A UfVX "fVgHz"UbX"&&"5df]" f&bXz	%ž&
Hi hcf]U``k cf_'	% 1	%'A Um	%ž&
9ggUm			

@Yb[h\.FYgdcbgYghceiYghcbg

8 YHJ]`g. `Hk c`cb`]bY`hYghg`YUW\`k cfh\`&\$ı žk]```VY`\Y`X`]b`K YY_`(`UbX`K YY_`%\$`cZh\Y`gYa YghYf"'H\Y hYghg`k]```Wca df]gY`g\cfh`Ubgk Yf`ei Ygh]cbg`hc[Yh\Yf`k]h\`U``cb[Yf`fYgdcbgY`cb`_Ymh\Ya Yg`UbX]bgh]hi h]cbg`Wcj YfYX`]b`h\Y`Wci fgY"'9UW\`hYghk]```VY`a Uf_YX`k]h\]b`&`k YY_g`cZgi Va]gg]cb`UbX`mci `k]``fYWY]j Y`bi a Yf]WU`[fUXYg"

5XX]h]cbU`XYhU]`g.

Mci 'UfY'fYei]fYX'hc'Xc'h\Y'Ugg][bYX'fYUX]b[g'Yj Yfmk YY_'Ug'h\Y'VUg]g'Zcf'hi hcf]U'`X]gW gg]cb"'H\YgY fYUX]b[g'U`gc'Zcfa 'h\Y'VUg]g'Zcf'cb`]bY'ei]nnYgž']b'K YY_'('UbX'K YY_'%\$"'H\Y'ei]nnYg'fYei]fY'mci 'hc XYa cbghfUhY'h\Uhmci '\Uj Y'fYUX'UbX'hf]YX'hc'i bXYfghUbX'h\Y'Ugg][bYX'dUggU[Ygž'Ug'k Y``'Ug'`YWh fY'UbX h' hcf]U``a UhYf]U`"'9UW\'hYghk]``'VY'a Uf_YX'k]h\]b'&'k YY_g'cZgi Va]gg]cb'UbX'ghi XYbhg'k]``'fYWY]j Y bi a Yf]WU`'[fUXYg"

 $\label{eq:higher} \mbox{Hi fb]h]b"gYh]b[."H\]g"Ugg][ba Ybh]g"gi Va]hhYX"h\fci [\ \ 'Hi fb]h]b"UbX"gh XYbhg"Xc"bchgYY"Hi fb]h]b q]a]`Uf]hmfYdcfhg"$

5ggYgga Ybh& Hi hcf]U`k cf_

GHUfhXUhY. Bch5dd`]WUV`Y

@Yb[h\.') \$\$ k cfXg

8 YHJ]`g. Mci ik]``VY 'UggYggYX 'cb'mci f'kcf_']b'UbX 'Wcblf]Vi hjcb'hc'h hcf]U`gž']bWi X]b['U'fYZYWhj Y 'ZjbU`fYdcfhhc'VY 'gi Va]hHYX 'Uhh\Y 'YbX'cZgYa YghYf'' Mci ik]``VY 'UXj]gYX'cZh\Y'a Uf_]b['Wf]hYf]U']b'h\Y Wci fgY [i]XY 'UbX']b'``YWhi fYg''' H\YfY k]```VY 'cddcfhi b]hjYg'h\fci [\ci hh\Y'gYa YghYf'hc'X]gW gg'mci f'dfc[fYgg']b WUgg'X]gW gg]cb'UbX'hc'VY'dfcj [XYX'U'dfcj]g]cbU`'UggYgga Ybh'[fUXY'UbX'Wca a Ybh'

5XX]h]cbU`XYhU]`g.

C by czh y U]a g c zh kcf]U g]g k y Vcd mci f eba] W Vca dyh bW Ubx g_] g h Uh [i a ybh Ubx x]gW gg]cb k]h ch y fg cb h y dfc v y a g k y Vcbg]xyf" Mci f dUfh]W]dUh]cb k] Vy UddfU]gy x UWVcfx]b['hc h y fy [i `Uf]miUbx ei U`]mc zmci f dUfh]W]dUh]cb z f h kcf]U gz Ubx mci f k] b [b y gg hc Vcbhf]vi h h hc x]gW gg]cbg k]h mci f z v ck gh xybhg" 5 g dUfhc zh y Uggy gga ybh z f h hcf]U dUfh]W]dUh]cb z mci 'Ufy 'f yei]fy x hc k f]h 'U g cfh fy dcfh cb mci f Wcbhf]vi h cb hc h hcf]U x]gW gg]cb 'Ubx Uf[i a ybh 'h y fy dcfh g ci `x vy 'Uddfcl]a Uh 'm) \$\$ k cfxg]b `Yb[h x z Ubx]g hc vy `Ubxyx]b vm% A Um = g \U``H _Y mci f f ydcfh]bhc 'Wcbg]xyfUh]cb]b `Ugg][b]b[mci 'U a Uf _ Zcf h hcf]U `dUfh]W]dUh]cb" b `Yugy bch h Uh mci 'W b cb m]a dfcj y mci f a Uf _ h fci [\ mci f f ydcfh z vi h h Uh mci k]``bch vy 'Ugg][by x 'U a Uf _ Zcf h hcf]U `dUfh]V]dUh]cb b b Ygg mci 'Xc `Ubx]b `Uffydcfh'

UddfYWJUhY 'h\ UhX]ZZYfYbhighi XYbhg`VY[]b`Zfca 'X]ZZYfYbh`Yj Y`g`cZi bXYfghUbX]b['UbX`WcbZJXYbWY"'H\YfY`]g j Uf]UhJcb`]b'h\Y`Zcfa 'cZhi hcf]U`g`Zfca 'k YY_fc 'k YY_ž'UbX`a cghighi XYbhg`ZJbX'h\Uhih\YmUfY`UV`Y`hc dUfhJWJdUhY`YZZYWhJ Y`m]b'ga U``![fci d`X]gW gg]cbg`]b'dUfhJW``Uf"

Mci 'a][\h'Z\bX']hi gYZ\`'\c'_YYd'U'k YY_\mX]Ufm'UbX'\c'XfUk 'cb'\h\Uh\Ug'\h\Y'VUg]g'cZmci f'Z\bU\`'fYdcfh'cb mci f'h \cf]\U\`dUfh\W\dh\cb''Gh XYb\g'k]\``VY'UXj]gYX'cZh\Y'a Uf_]b['W']\h'Yf\U'\b' \fyg''[i]XY'UbX']b\\Y\Wfi fYg'''H\YfY'k]\``VY'cddcfh b]\h\Yg'\h\fci [\ci h\h\Y'gYa Yg\h'f'\c'X]gW gg'\h\Y]f'dfc[fYgg']b'WUgg X]gW gg]cb'UbX'\c'VY'dfcj]XYX'U'dfcj]g]cbU``UggYgga Ybh'[fUXY'UbX'\c'a a Ybh'

 $\label{thm:limit} \begin{tabular}{ll} Hi & fb & fb & gYh &$

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GhUfh'XUhY.

8YHJ]`q.

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7ci fgY'GWXYXi `Y

J]Yk 'WUgg'hja YhUV'Y

H]a YHJV`Y

K YY_'%'%': YVfi Ufm!'&(: YVfi Ufm	· VhVfII']hhfcVi MNch 'C f]VhHINch IIhV
	; YbYfU``]blfcXi Wi]cb.`Cf]YblUh]cb'UbX ei Ygh]cbg"'H\Y``YWi fY`]b'h\]g'k YY_'Wcj Yfg cf[Ub]gUh]cbU`a UhYfg']bWi X]b['h\Y'Wci fgY'ci h]bYz cVYWi]j Yg'UbX'cf[Ub]nUh]cb"'5ggYgga Ybh fYei]fYa Ybhg'k]```VY`X]gW ggYX']b'XYhU]`"'Hi hcf]U` ha Yg'k]```VY`WcbZ]fa YXz'UbX'cZ[WY'\ci fg Ubbci bWYX"'H\Y``YWi fY'U`gc'dfcj]XYg'Ub'cj Yfj]Yk cZh\Y'5i ghfU`]Ub'dc`]hJWJ``gmghYa 'UbX']hg'df]bWJd`Yg"' Hi hcf]U`g'VY[]b']b'K YY_''%"'I gYZ ``VUW_[fci bX fYUX]b[']bWi XYg. • H\Y'5i ghfU`]Ub'7cbghhi h]cb
K YY_'&.'&) ': YVfi Ufm!'' A UfVX	H\Y'5i glfU']Ub Wcbgh]hi h]cbU'ZfUa Yk cf 8Ya cWfUWmUbX'fYdfYgYbhUh]cb"'H\Y 'YWh fY YI d'U]bg'h\Y'glfi Wh fY UbX'df]bW]d'Yg'cZh\Y 5i glfU']Ub Wcbgh]hi h]cbz'UbX'h\Y'K Ygha]bghYf gmghYa 'cZfYgdcbg]V]']mm'H\Y'j Uf]ci g'mm'Yg'cZ dUf']Ua YbhUfmcZ]WYg'UbX'dfcWYXi fYg'UfY'YI d'U]bYX]b'fY'Uh]cb'rc'h\Y'GYbUhY'UbX'h\Y'< ci gY'Ug fYdfYgYbhUh] Y'fUbX'XYa cWfUh]WE']bgh]hi h]cbg"'H\Y 'YWh fY'U'gc'YI d'U]bg'h\Y]f'fY'Uh]cb'rc'j Uf]ci g h\Ycf]Yg'cZfYdfYgYbhUh]cb'z]bWi X]b['Ui h\cf]gUh]cbz' XY'Y[Uh]cbz'fYdfYgYbhUh]cb'UbX'h\Y'WcbhYghUV'Y']XYU cZ'a UbXUhY''' Mci 'g\ci 'X'VY'UV'Y'rc'ci h]bY'h\Y k Umg']b'k\]W\'h\Y'5i glfU']Ub'Wcbgh]hi h]cb'X]glf]Vi hYg dck Yf'VYlk YYb'h\Y'Ufa g'cZ[cj Yfba YbhUbX'']a]hg h\Y'cj YfU'`dck Yf'cZ[cj Yfba Ybh' • 5'Ub': YbbUz'H\Y'5i glfU']Ub'GmghYa 'cZ ; cj Yfba YbhZ'7\UdhYf'&cZ9 ahWd'_Wf\$`V Ba'[f]&e['3gefc&{S'\%h\'YX''zYg\E'5`Ub : YbbUz'>UbY'F cVV]bg'UbX'>c\b'Gi a a Yfg fGmXbYm'DYUfgcb'5i glfU']Ubg'7 ci 'Xb'h • GhYd\Yb' <uff]b['hcbz'5i 'xb'h<="" ci="" glfu']ubg'7="" td=""></uff]b['hcbz'5i>

	6f]Ub`7cgtUfž'H\Y'9`YWtcfU``GmgtYa'ž 7\UdfYf'-`cZ9ahWt_WfS`VBa{ff[Ue[` 3geft\${S'%\$t\YX"žYX"'5`Ub`: YbbUž'>UbY FcVV]bg`UbX`>c\b`GiaaYfg`fGmXbYm DYUfgcb`5igtfU`]Už&\$%'Ł

FYgci fW/g

DfYgWf]VYX F Ygci fWYg

• ; cj Yfba YbhUbX Dc`]hJWg]b 5i glfU`]U

FYWca a YbXYX FYgci fW/g

Hi hcf]U`g`dfYgi ddcgY`h\Uhmci `\Uj Y`fYUX`UbX`a UXY`Ub`\cbYghUhhYa dh'hc`i bXYfghUbX`h\Y`fYei]fYX fYUX]b['Zcf`YUW\`k YY ž'k \]W\`]g`qhUffYX`]b`h\Y`Wci fqY`[i]XY`VY`ck "

Zcfa U``m'Wca d`YhYXž'UbX'ghi XYbhg'UfY'U`gc'U`k Umg'k Y`Wca Y`hc'WcbhUWhih\Y'Wci fgY'Wcbj Ybcf'UZhYf'WUggYg\Uj Y'Z]b]g\YXžYj Yb'a Ubm'mYUfg'UZhYf'h\Y'Wci fgY'\Ug'YbXYX"'5```ZYYXVUW_']g'WUfYZ\``m'Wcbg]XYfYXžfYZYWhYX'i dcbžUbX'i gYX'hc'Yj U`i UhY'UbX'fYZ]bY'U```UgdYWhg'cZh\Y'Wci fgY''

Gi Va]gg]cb cZ5ggYgga YbhHUg_g

Gi Va]gg]cb cZ5ggYgga YbhHUg_g

Hi fb]h]b Gi Va]gg]cb

_Zmci YbWci bhYf'U'dfcV`Ya 'k \Yb'UhYa dhJb['hc'gi Va]hmci f'Ugg][ba Ybhh\fci [\ 'Hi fb]hJbž'd`YUgY
hY`Yd\cbY'9| hYfbU``Gi ddcfhcb'-',) '' ' ' %cf'Ya U]`'h\Ya 'cb'Y! hYfbU`hY`hgi ddcfh4 i bgk "YXi "Ui '"'Gi ddcfh
\ci fg'UfY', .\$\$Ua '-'%\$.\$\$da 'cb'k YY_XUmg'UbX'-.\$\$Ua '-') .\$\$da 'cb'k YY_YbXg'fl *) 'XUmg'U'mYUft'' =Zmci
UfY'i bUV`Y'hc'gi Va]hmci f'Ugg][ba YbhXi Y'hc'U'ZUi 'hk]h\'Hi fb]hJb'mci 'a Um'Udd'mZcf'Ub'Y! hYbg]cbžVi hmci
a i ghfYhU]b'mci f'hJW_Yhbi a VYf'Zfca '9! hYfbU``Gi ddcfhfU'cb['k]h\'Ubmch\Yf'fY`Yj UbhXcW a Ybhgt'hc
]bWi XY'Ug'Yj]XYbW'hc'gi ddcfhmci f'Y! hYbg]cb'Udd`]WUhJcb'' =Zmci 'Ya U]`'9! hYfbU``Gi ddcfhmci 'k]``

5 WUXYa]W<cbYghmUbX D`U[]Uf]ga

D`U[]Uf]ga `]g`i g]b['h\Y`k cfXg`cf`]XYUg`cZch\Yfg`UbX`dfYgYbh]b['h\Ya `Ug`mci f`ck b"'\lefi\Wb'hU_Y`a Ubm Zcfa gžZfca `XY`]VYfUhY`W\YUh]b['hc`UVW]XYbhU`mWcdm]b['Zfca `U`gci f\W' k]h\ci hUW_bck `YX[Ya Ybh'

IBGK [fcidgdu[]Uf]ga]bhch\YZc``ck]b[WUhY[cf]Yg.

7 cdm]b[.'i g]b['h\Y'gUa Y'cf'j Yfmg]a]`Uf'k cfXg'hc'h\Y'cf][]bU`hYl h'cf']XYU'k]h\ci h'UW_bck `YX[]b['h\Y gci fW'cf'i g]b['ei chUh]cb'a Uf_g"'H\]g'U`gc'Udd`]Yg'hc']a U[Ygž'Ufh'UbX'XYg][b'dfcYWhgž'Ug'k Y``'Ug dfYgYbhUh]cbg'k\YfY'gca YcbY'dfYgYbhg'Ubch\Yf'g']XYUg'cf'k cfXg'k]h\ci h'WfYX]h'

⇒ Uddfcdf]UhY 'dUfUd\fUg]b[...7\Ub[]b['U'ZYk 'k cfXg'UbX'd\fUgYg'k \]'Y 'a cghmfYhJ]b]b['h\Y'cf][]bU` ghfi Wh fY 'UbX#cf'dfc[fYgg]cb'cZ]XYUg'cZh\Y'cf][]bU'žUbX']bZcfa Uh]cb'k]h\ci hUW_bck `YX[Ya Ybh'H\]g'U`gc Udd`]Yg']b'dfYgYbhUh]cbg'k \YfY'gca YcbY'dUfUd\fUgYg'Ubch\Yf'g']XYUg'cf'k cfXg'k]h\ci hWfYX]hUbX'hc d]YW]b['hc[Yh\Yf'ei chYg'UbX'dUfUd\fUgYg']bhc'U'bYk 'k \c`Yžk]h\ci hUddfcdf]UhY'fYZYfYbW]b["

7 c``i g]cb. 'k cf_]b['k]th 'cth Yfg'Vi hidUgg]b['cZZth Y 'k cf_'Ug'U'dYfgcb'g']bX]j]Xi U``k cf_"'7 c``i g]cb'U`gc]bWi XYg'dfcj]X]b['mci f'k cf_'thc'Ubcth Yf'ghi XYbh'VYZcfY'th Y'Xi Y'XUHYZcf'Zcf'th Y'di fdcgY'cZth Ya

- i bXYfglUbX`mci f`f][\hg`UbX`fYgdcbg]V]`]h]Yg`Ug`U`gh XYbh'Uhi BGK
- VY'Uk UfY'cZd`U[]Uf]ga ž'Wcdmf][\hzil BGK 'Gh XYbh7cXY'cZ7cbXi WhUbX'5WWdhUV`Y'I gY'cZ I BGK '= H'F Ygci fWfg'Dc`]Wh
- VY 'Uk UfY 'cZh\Y 'ghUbXUfXg 'cZVY\Uj]ci f'Yl dYWhYX 'cZYj YfmcbY ']b 'h\Y 'l BGK 'Wcaaib]hm
- `cWUHY 'gYfj]WYg'UbX `]bZcfa UH]cb'UVci hI BGK 'UbX'I BGK '@]VfUfm

5 WUXYa]W = b Zcfa Uh cb ·

: cf`YggYbhJU``gh XYbh`]bZcfa UhJcb`fY`UhJb[hc.

- fYei Yghg Zcf Yl hYbg]cb/
- `UhY`gi Va]gg]cbg`[i]XY`]bYg/
- fYj]Yk 'cZa Uf_g/
- I BGK '< YU`h\ 'UbX'GUZYhm'dc`]W[Yg/
- YI Ua]bUhcb dfcWXi fYg/
- gdYWJU``Wcbg]XYfUHjcb`]b`H\Y`Yj YbhcZ]``bYgg`cf`a]gUXj Ybhi fY/
- gh XYbhYei]lmUbX X]gUV]`]lm/
- UbX'ch\Yf'YggYbhJU`'UWUXYa]W]bZcfa UhJcbž'gYY

\hdg.##k k k "Uffig"i bgk "YXi "Ui #W ffYbhgh XYbhg#UWUXYa]W]bZcfa Uh]cb#dfchcWc`g![i]XY`]bYg#
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a U[Y'7fYX]h

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7F = 7CG

7F = 7CG Dfcj]XYf 7cXY. \$\$\$-,;