MATHEMATICS ENRICHMENT CLUB.[1]
Problem Sheet 3, May 21, 2013

1. The perimeter of a base of a rectangular brick with integer sides is 18 cm, whilst its volume is 42 cm^3. What is its height?

2. Calculate
$$1 \quad \frac{1}{2} \quad 1 \quad \frac{1}{3} \quad 1 \quad \frac{1}{4} \quad \therefore \quad 1 \quad \frac{1}{2008} \quad \vdots$$

3. Find the smallest positive integer whose square ends in (a) 09 and (b) 9009.

4. Show that if $a; b$ are positive numbers such that $ab \quad 1$ then
$$\frac{a}{b+1} + \frac{b}{a+1} + (1 \quad a)(1 \quad b) \quad 1:$$

5. Suppose we have the numbers $x_0 = 0; x_1 = 1$ and $x_{n+1} = x_n + 2x_{n\ 1}$ for $n \quad 2$.

   (a) Write down the numbers x_n for $n = 2; 3; 4; 5; 6$.

   (b) Show that there is no n for which $x_n = 1999$. (Hint: Use modulo 8 arithmetic).

   (c) Show that $x_n = \frac{2^n \quad (\ 1)^n}{3}$ satis es the equation.

6. In ABC, extend the sides AB and AC and draw a circle outside the triangle which touches BC and these two produced sides. This circle is called the *escribed circle* of the triangle.

   (a) Show that $r_1 = \frac{A}{s\ a}$, where r_1 is the radius of the escribed circle, A is the area of ABC, a is the length of BC and s is half the perimeter of ABC.

   (b) Show that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}r$ is the radius of the incircle (recall last weeks result.)

7. $ABCD$ is a parallelogram, Q a point inside it. Prove that the sum of the areas of AQB and CQD is half the area of $ABCD$.

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[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

Senior Questions

1. Prove that the square of the nth triangle number is the sum of the first n cubes, i.e.

$$\left(\sum_{k=1}^{n} k\right)^2 = \sum_{k=1}^{n} k^3; \quad \text{for } n \geq 1.$$

2. Find the limit $\lim_{n \to \infty} \dfrac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3}$.

3. A hand of eight cards is dealt from a standard pack. How many hands contain exactly three cards of the same value and the remaining cards from the remaining suit?