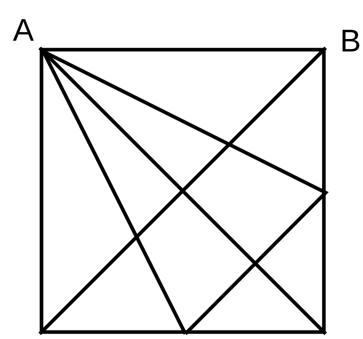
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 2, May 5, 2015[1]

1. Solve
$$\frac{y+x}{\sqrt{y}+1} = x.$$

2. Let x, y and z be integers. Show that if $x - y + 2z$ is divisible by 11, then so is $12x + y - 13z$.

3. Anna and Boris move simultaneously towards each other, from points A and B respectively. Their speeds are constant, but not necessarily equal. Had Anna started 30 minutes earlier, they would have met 2 kilometers nearer to B. Had Boris started 30 minutes earlier instead, they would have met d kilometers nearer to A. Find d.

4. A triangle APQ is drawn inside a square, such that the points P, Q are on the sides BC and DC of the square $ABDC$, with the length of PC and QC equal. Draw a line from P parallel to AC, to intersect the diagonal DB at the point X as shown below.



   (a) Show that the triangle PXB is isosceles.

   (b) Show that the perimeter of APQ can not be more than the perimeter of ABD.

5. A four digit number and its square ends in the same four digits. Find the number.

[1] Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

6. A 3×3 magic square is a grid filled with the numbers 1 to 9 so that the sum of rows, column and diagonal are all equal. E.g

6	1	8
7	5	3
2	9	4

Counting different orientations of the grid as the same magic square, prove that the above example is the only solution.

Senior Questions

The First two problems are based on polynomials: A polynomial of degree k is a function of the form $P_k(x) = a_0 + a_1x^1 + a_2x^2 + \dots + a_kx^k$, where $a_0, a_1, a_2, \dots, a_k$ are real numbers. For example $P_2(x) = 5x^2 + 3x + 1$ is a polynomial of degree 2, with $a_0 = 1$, $a_1 = 3$ and $a_2 = 5$. Also $P_2(x^2) = 5(x^2)^2 + 3x^2 + 1$ is a polynomial of degree 4.

1. Let $P_3(x)$, $Q_2(x)$ and $R_3(x)$ be polynomials of x, show that
   (a) $P_3(x) \cdot Q_2(x) + R_3(x^2)$ is a polynomial of degree 6.
   (b) $P_2(Q_3(\sqrt{x}))$ is a polynomial of degree 3.
2. Let $f(x) = \exp(\frac{1}{x})$, and let $f^{(k)}(x)$ denote the k^{th} derivative of f with respect to x. For $k \geq 2$, use induction to show that
$$f^{(k)}(x) = P_{2k}\left(\frac{1}{x}\right)\exp\left(\frac{1}{x}\right).$$
3. [2] Use induction to show that
$$\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \leq \frac{2}{3}n\sqrt{n}.$$

[2] This problem provided by Adam Solomon.