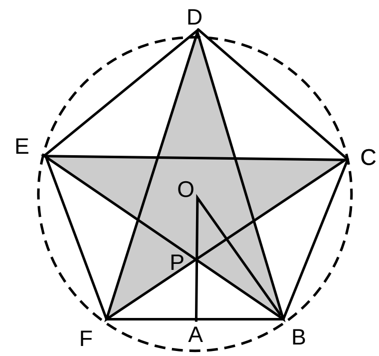
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 5, May 26, 2015 [1]

1. (a) Note that if a number has remainder 1 when divided by 7, then any powers of this number divided by 7 still has remainder 1. Since $2^{2015526} = 2^{3 \cdot 671842} = 8^{671842}$ and 8 has remainder 1 when divided by 7, we can conclude that $2^{2015526}$ has remainder 1 when it is divided by 7.

   (b) Since $2^5 = 32$, the last digit of 2^{5^n} is 2 for $n = 1, 2, \dots$. Now because all we care about is the last digit of 2^{2015}, we can simplify the problem by factoring $2^{2015} = (2^{5^3})^{16}(2^5)^3 = (2^{5^4})^3(2^{5^3})(2^5)^3$. Because we already know that the last digit of each 2^{5^n}, $n = 4, 3, 1$ is 2, we can find the last digit of 2^{2015} by considering the last digit of $2^3 \cdot 2 \cdot 2^3$, which is 2.

2. Geralt can roll either $\{1, 2, 3, 4, 5\}$ or $\{2, 3, 4, 5, 6\}$. The number of ways to obtain either of these combination is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$, and the total number of outcomes of rolling 5 dices is 6^5



4. Let O be the centre of the pentagon, A is the point of intersection between the bisector

6. (example by Madeleine. K) The last digit of x must be 0, because we need an integer after increasing x by 10%. Now in order to decrease the sum of digits of a number after increasing it by 10%, we look for a number that has a lot of digit that will "carry" up when multiplied by 1.1; an example would be a number starting with m lots of 3's, with n lots of 6's in the middle and a 0 at the end, because increasing this number by 10% gives a number with the $n-1$ lots of 3's, $m-1$ lots of 6's, one of each 7, 2 and 0. So the equation we need to solve is $(n-1)3 + (m-1)6 + 9 = 0.9(3m + 6n)$; $n = m = 10$ is a solution.

Senior Questions

1. The trick is to apply a change of variable, so that the two graphs become symmetrical. Let $X = x/10$ and $Y = 10y$, then $Y = 10\cos(10X)$ and $X = 10\cos(10Y)$. Let A be the sum of the new X-coordinate, and B the sum of the new Y-coordinate, then because the graph of X and Y are symmetrical, we have that $\frac{A}{B} = 1$. Now using the fact that the coordinates are positive, we have $A = a/10$ and $B = 10b$, therefore $\frac{a}{b} = 100$.

2. Apply a change of base on the logarithm.

3. First we consider case $0 < x \leq 1$. The RHS of $x = \frac{1}{2}\left(y + \frac{1}{y}\right)$ is the average of y and $1/y$, thus $y \geq x \geq \frac{1}{y}$ and $y \geq \frac{1}{x} \geq \frac{1}{y}$. Similarly, $z \geq y, \frac{1}{y} \geq \frac{1}{z}$, $t \geq z, \frac{1}{z} \geq 1/t$ and $x \geq t, \frac{1}{t} \geq 1/x$. From this we conclude that $x \geq t \geq z \geq y \geq x \geq \frac{1}{x} \geq \frac{1}{y} \geq \frac{1}{z} \geq \frac{1}{t} \geq \frac{1}{x}$, so the only solution for this case is $x = y = z = t = 1$.

   Using the same arguments as above, we can deduct that there is no solution for $x > 1$ and for the case $x < 0$, we have $x = y = z = t = -1$.

   To extend to the 2015 variable case, note that the above arguments does not depend on the number of variables we had initially.