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MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 13, August 15, 2016

1. Consider

$$\begin{aligned} f(x) &= (1+x)(1+x^2)(1+x^4)(1+x^8)\ldots \\ &= 1 + x + x^2 + x^3 + x^4 \ldots + \\ &= \frac{1}{1-x}, \end{aligned}$$

where last line is due to the sum of an infinite geometric sequence. Hence, setting $x = 1/2^2$ in $f(x)$, we have

$$f\ \frac{1}{}$$

4. Since $2^x = 6^{-z}$, we have

$$2 = 6^{-\frac{z}{x}}. \qquad (1)$$

Similarly, since $3^y = 6^{-z}$, we have

$$3 = 6^{-\frac{z}{y}}. \qquad (2)$$

Therefore, combining (1) and (2), we have

$$6 = 2 \times 3 = 6^{-\frac{z}{x}} \times 6^{-\frac{z}{y}} = 6^{-\frac{z}{x}-\frac{z}{y}}.$$

In particular,

$$1 = -\frac{z}{x} - \frac{z}{y},$$

so that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$$

5. The solution is 89. This can be obtain by using binomial expansion carefully. Alternatively, note that

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{11} + \left(\frac{1-\sqrt{5}}{2}\right)^{11}}{\sqrt{5}}.$$

is the 11^{th} term of the Fibonacci number, see https://en.wikipedia.org/wiki/Fibonacci_number or Question sheet 6, 2016.

6. Let x the number of dollars and y the number of cents on the cheque. Note that three times the value of the cheque must be less than \$100.22, which implies $x < 34$. Now, we can write the value of the cheque as $100x + y$ cents, then the amount the bankers gave out was $3(100x + y) - 22$ cents. Therefore,

$$\begin{aligned} 100y + x &= 3(100x + y) - 22 \\ 97y &= 299x - 22 \\ 97(y - 3x) &= 8x - 22. \end{aligned}$$

Hence, using $x < 34$

$$97(y - 3x) = 8x - 22 \leq 250. \qquad (3)$$

The LHS equality of (3) implies $y - 3x$ must be even. The RHS inequality implies $y - 3x \leq 2$. From this, we conclude that

$$\begin{aligned} y - 3x &= 2 \\ 97 \times 2 &= 8x - 22. \end{aligned}$$

Solving the above system simultaneously yields $x = 87$ and $y = 27$.

Senior Questions

1. Let $f(x)$ denote the number of consecutive primes between x and $x + 2015$. Clearly $f(1) > 15$. Moreover, for consecutive inputs x and $x + 1$, the function f can only vary by $0; 1$ or 1; i.e $f(x)$ di ers to $f(x$