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MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 7, June 13, 2016

1. Let X be the total age of the committee members 4 years ago. Since there are 10 committee members, the total age of the members will increase by 4×10 in 4 years time; that is, the total age of the committee members currently is $X + 40$. Now, since only one member is replaced, the new member must be 40 years younger than the member being replaced.

2. Expanding and simplifying the expression $(1+k)^3 - k^3$, we have

$$\begin{aligned}(1+k)^3 - k^3 &= (1 + 3k + 3k^2 + k^3) - k^3\\ &= 1 + 3k + 3k^2.\end{aligned}$$

Hence,

$$\begin{aligned}3k^2 &= (1+k)^3 - k^3 - 3k - 1\\ k^2 &= \frac{1}{3}[(1+k)^3 - k^3 - 3k - 1]. \qquad (1)\end{aligned}$$

Therefore, we need to sum the RHS of (1) from $k = 1$ to $k = n$. Consider the sum of $[(1+k)^3 - k^3]$ from $k = 1$ to $k = n$, one has

$$\begin{aligned}&(2^3 - 1^3) + (3^3 - 2^3) + \ldots + [n^3 - (n-1)^3] + [(n+1)^3 - n^3]\\ = &-1^3 + (2^3 - 2^3) + (3^3 - 3^3) + \ldots + [(n-1)^3 - (n-1)^3] + [(n)^3 - (n)^3] + (n+1)^3\\ = &(n+1)^3 - 1.\end{aligned}$$

Which implies

$$\begin{aligned}[(n+1)^3 - 1] - 3(1 + 2 + \ldots + n) - n &= [(n+1)^3 - 1] - 3\frac{n(n+1)}{2} - n\\ &= (n+1)[(n+1)^2 - 3\frac{n}{2} - 1]\\ &= n(n+1)(n+1/2). \qquad (2)\end{aligned}$$

Therefore, by (1) and (2)

$$1^2 + 2^2 + 3^2 + \ldots + n^3 = \frac{n(n+1)(2n+1)}{6}.$$

1.

the last digit o $\sum_{m=0}^{9} r^{14}$, we note that for any integer k, one has $k^5 = k \mod 10$, thus $k^{13} = (k^5)^2 \quad k^3 = k^5 = k \mod 10$, so that $k^{14} = k^2 \mod 10$. Therefore

$$\sum_{r=0}^{9} = r^{14} = \frac{9 \quad 10 \quad 119}{6} = 285 = 5 \mod 10;$$

where we have used the results of $Q2$ to obtain the second equality. Hence, the last two digits of $\sum_{m=0}^{9} r^{14}$ is 50.

3. The number of towns is nite and there are only six ways of entering and leaving any given town. Hence the traveller must eventually pass through some town T twice entering and leaving in the same direction. From this point on the traveller loops forever.

   Suppose that after two loops the traveller turns around and retraces his steps turning left where he turned right and vice versa. This reverse the journey staying on the loop hence the initial town is on the loop at leat once and up to six times.