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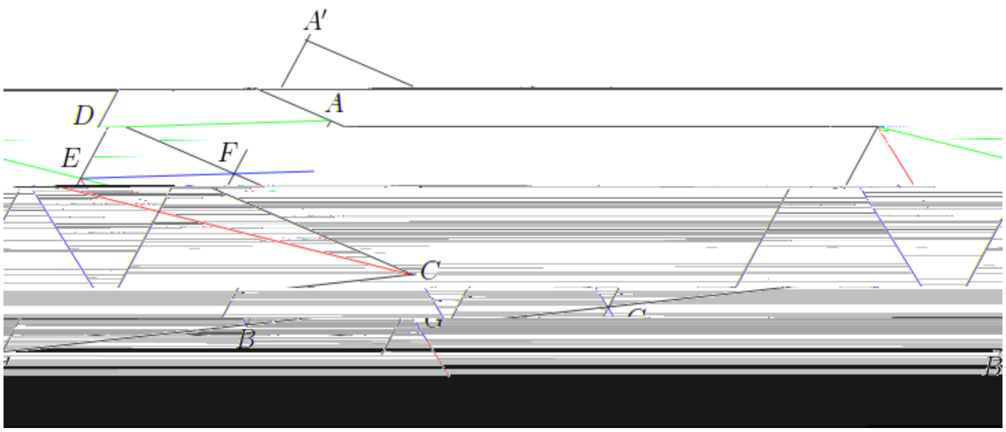
MATHEMATICS ENRICHMENT CLUB.
Solution Sheet ?, July 24, 2017

1. Note that $90 \quad n = 2 \quad 3^2 \quad 5 \quad n$, so if the product is a cube in particular must have 2^3, 3^3 and 5^3 as divisors. Thus, it su ces to choose $n = 3 \quad 2^2 \quad 5^2 = 300$, so we have

$$90 \quad n = 2^3 \quad 3^3 \quad 5^3 = (2 \quad 3 \quad 5)^3:$$

2. The result is clear if the line drawn is either of the diagonals, for they are lines of symmetry. Suppose now that the line drawn is not a diagonal. With the diagonals also drawn, the parallelogram is divided into 6 triangles. Well leave the rest up to you, but x^2 x^3

5. Begin by constructing the equilateral triangle ADB. Draw the line CD to intersect AB at E. Draw EG parallel to DB and EF parallel to DA. Connect F and G, then the triangle EFG is equilateral. To prove this is true, construct $B'A'$ parallel to BA and passing through D, and show that $BB'D$ and $AA'D$ are similar to GBE and FAE respectively.



6. Let us write the elements in A as $a_1, \dots, a_k$, with

$$a_1 < a_2 < \cdots < a_{k-1} < a_k.$$

(a) Note that we can construct the chain

$$a_1 + a_1 < a_1 + a_2 < a_1 + a_3 < \cdots < a_1 + a_k < a_2 + a_k < \cdots < a_{k-1} + a_k < a_k + a_k,$$

of elements in $A + A$ that has $2k - 1 = 2|A| - 1$ distinct elements.

(b) If $|A + A| = 2|A| - 1$ this implies that the elements in the chain are all the possible ones. Thus, we can construct the following chains

$$\begin{aligned} a_1 + a_1 < a_1 + a_2 < a_1 + a_3 < a_1 + a_4 < \cdots < a_1 + a_k < a_2 + a_k < \cdots \\ a_1 + a_1 < a_2 + a_1 < a_2 + a_2 < a_2 + a_3 < \cdots < a_2 + a_{k-1} < a_2 + a_k < \cdots \end{aligned}$$

of length $2|A| - 1$. Since element by element, both chains must coincide, in particular we have that for every $i = 1, \dots, k - 1$

$$a_2 + a_i = a_1 + a_{i+1}$$

which implies in particular that

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = a_k - a_{k-1} = d$$

or in other words: A is an arithmetic progression of difference d.

Senior Questions

1. We will resolve it for $n = 5$ and leave the generalisation to the reader. We will show how to cut a square into 5 pieces of the same area.

   To do so, we will divide each side of the square in 5 equal segments (dividing the whole perimeter in 20 equal parts, represented in the gure by black and red circles). Now, we will join the center of the square with every fourth vertex (coloured in red) and obtain 5 pieces with the same area.