[redacted]

MATHEMATICS ENRICHMENT CLUB.

Solution Sheet 17, September 24, 2018

1. Firstly, we calculate the prime factorisation of N.

$$\begin{aligned} N &= 1^9 \times 2^8 \times 3^7 \times 4^6 \times 5^5 \times 6^4 \times 7^3 \times 8^2 \times 9^1 \\ &= 2^8 \times 3^7 \times 2^{12} \times 5^5 \times (2 \times 3)^4 \times 7^3 \times 2^6 \times 3^2 \\ &= 2^{30} \times 3^{13} \times 5^5 \times 7^3 \end{aligned}$$

A divisor of N that is a perfect square has a prime factorisation with all primes raised to an even number. Thus there are 16 choices for the number of 2's $(0, 2, 4, \ldots, 30)$; 7 choices for the number of 3's; 3 choices for the number of 5's; and 2 choices for the number of 7's. The total is $16 \times 7 \times 3 \times 2 = 672$.

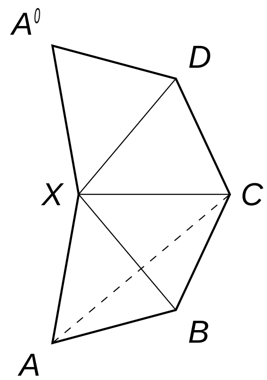
2. Firstly, note that neither b nor c can be zero. Then simplifying the given equation, we have

$$\begin{aligned} \frac{b}{b} - \frac{a/b}{c} &= \frac{a}{b/c} - \frac{c}{c} \\ \frac{a}{bc} &= \frac{ac}{b} \\ ab &= abc^2 \\ ab(1 - c^2) &= 0 \end{aligned}$$

This equation has the solutions $a = 0$, $b = 0$ or $c = \pm 1$. As noted previously, $b \neq 0$. So we are left with $a = 0$ or $c = \pm 1$.

We now calculate the number of triplets, making sure not to inadvertently double-count any. If $a = 0$, then there are 20 choices for b and c, as $b, c \neq 0$. Thus there are 400 triplets with $a = 0$. If $c = 1$, then there are 20 choices for a, as the $a = 0$ case has already been counted in the previous step, and also 20 choices for b. This gives us another 400 triplets. Similarly, if $c = -1$, then there are 400 triplets. Thus there are 1200 triplets altogether.

3. Imagine that we detach the pyramid from its base, cut along AX and spread the top of the pyramid out flat. We would get a (possibly convex) polygon like that shown below.



The shortest distance from A to C is the straight line shown by the dotted path. We know that $AB = BC = b$. Let the length of AX be s. Then $AX = BX = CX = s$ (all the faces of the pyramid are isosceles triangles). Let \ (allLe 11.9552 Tf 5.514 0 T9 Q j

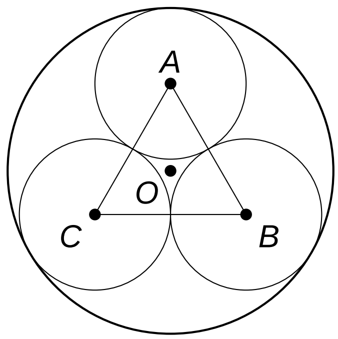
Thus

$$AC^2 = 4b^2\left(1 - \frac{b^2}{4h^2 + 2b^2}\right)$$

$$\Rightarrow AC = 2b\sqrt{1 - \frac{b^2}{4h^2 + 2b^2}}.$$

Since $\sqrt{1 - \frac{b^2}{4h^2+2b^2}} < 1$, we can see that the ant will always walk a shorte [redacted] it goes over the pyramid rather than around the base.

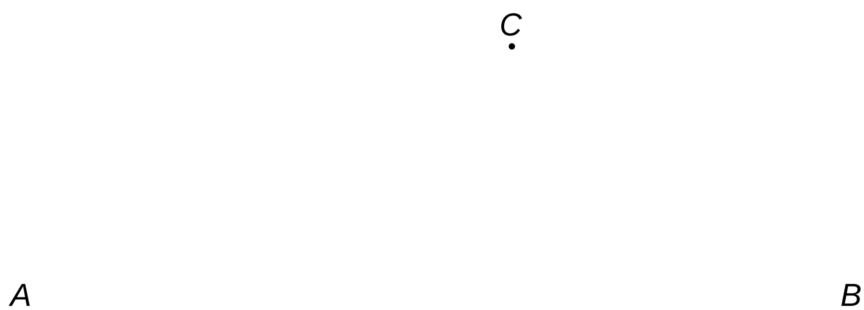
4. Let the centres of the small circles be A, B and C, and the centre of t [redacted] O.



Then $\triangle ABC$ is isosceles with side length $2r$, and meu8.2 [redacted] circumference is half the angle at the centre. So the vertex lies somewhere on the major arc.

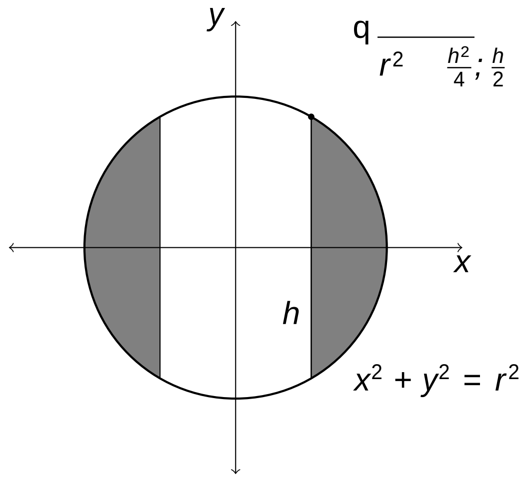
This construction fails if = 90 , as there is no unique point of intersection of the two rays *AE* and *BE*. However, in this case, *AB* is the diameter of *M* (as a consequence of Thales' theorem), and the construction is simple. If is obtuse, we can repeat the procedure outlined above, with the proviso that *P* will now lie on the minor arc of the circle. Some minor details of the proof will di er.

(b) Suppose that the angle at the vertex is , the altitude has height *h* and the median length *m*.



Senior Questions

1. We calculate the volume by integration using circular shells.



The volume, V is given by

$$V = 2\pi \int_a^b xy\,dx$$

In this case,

$$\begin{aligned} y &= \sqrt{r^2 - x^2} - \left(-\sqrt{r^2 - x^2}\right) \\ &= 2\sqrt{r^2 - x^2}. \end{aligned}$$

The limits of integration are $a = \sqrt{r^2 - \frac{h^2}{4}}$ (from Pythagoras' theorem) and $b = r$. Thus

$$\begin{aligned} V &= 2\pi \int_{\sqrt{r^2 - \frac{h^2}{4}}}^{r} 2x\sqrt{r^2 - x^2}\,dx \\ &= 2\pi \left[-\frac{2}{3}(r^2 - x^2)^{3/2}\right]_{\sqrt{r^2 - \frac{h^2}{4}}}^{r} \\ &= -\frac{4\pi}{3}\left[(r^2 - x^2)^{3/2}\right]_{\sqrt{r^2 - \frac{h^2}{4}}}^{r} \\ &= \frac{4\pi}{3}\left(\frac{h^2}{4}\right)^{3/2} \\ &= \frac{\pi h^3}{6} \end{aligned}$$

2. (a) Note that if $|x| < 1$ then the RHS is a convergent geometric series with $a = 1$ and common ratio $-x$. Thus

$$1 - x + x^2 - x^3 + \dots = \frac{1}{1 - (-x)} = \frac{1}{1 + x}$$