

2. (a) Note: this question should have read: "Explain why, if $a^2 + b^2$ has a fixed value, ab is greatest when $a = b$."

For all real numbers a and b ,

$$0 \leq (a - b)^2$$

$$0 \leq a^2 - 2ab + b^2$$

$$ab \leq \frac{a^2 + b^2}{2}$$

Furthermore, equality only holds if $a = b$.

(b)

$$x^4 + y^4 = x^4 + 2x^2y^2 + y^4 - 2x^2y^2$$

$$= (x^2 + y^2)^2 - 2x^2y^2$$

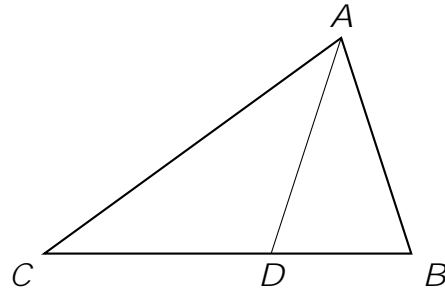
$$= c^4 - 2(xy)^2$$

This quantity is minimised when xy is maximised. That is, when $x = y$. If $x = y$, then $x^2 = y^2 = \frac{c^2}{2}$, in which case

$$x^4 + y^4 = c^4 - 2 \left(\frac{c^2}{2}\right)^2 = \frac{c^4}{2}$$

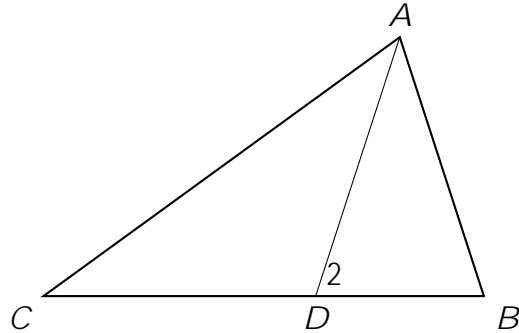
3. Suppose that ABC is a triangle. AD is an angle bisector, and $\triangle ADB$ and $\triangle ADC$ are both isosceles. Let $\angle CAD = \alpha$. Then $\angle DAB = \alpha$. Consider $\triangle ACD$, which is isosceles. There are three possibilities:

- (a) $\angle ACD$ is the vertex;
- (b) $\angle ADC$ is the vertex; or
- (c) $\angle CAD$ is the vertex.



If $\angle ACD$ is the vertex then $\angle CAD = \angle ACD = \alpha$ and thus $\angle ACB = 180^\circ - 2\alpha$.

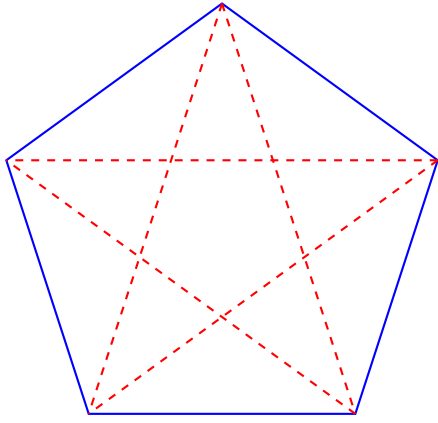
If $\angle ADC$ is the vertex then $\angle ACD = \angle DAC = \alpha$ and thus $\angle ADB = \angle ACD + \angle CAD$, since it is the external angle of $\triangle ADC$. So $\angle ADB = 2\alpha$. Since $\triangle ADB$ is also isosceles, it also has a pair of equal angles, and so $\angle ABD = \alpha$ or $\angle ABD = 2\alpha$.



If $\angle ABD = \alpha$, then by the angle sum of $\triangle ABD$, we have $4\alpha = 180^\circ$, so $\alpha = 45^\circ$. Hence $\angle A = 90^\circ$, $\angle B = \angle C = 45^\circ$. If, however, $\angle ABD = 2\alpha$, then by the angle sum of $\triangle ABD$, we have $5\alpha = 180^\circ$, so $\alpha = 36^\circ$. Hence $\angle A = \angle B = 72^\circ$, $\angle C = 36^\circ$.

If $\angle CAD$ is the vertex, then we have two cases that are essentially the same as those above, only with the orientations of the smaller triangles reversed. Hence we get the same two solutions.

4.4.



But $b = \frac{x}{\cos}$ and $a = \frac{c - x}{\cos 2}$, so

$$\frac{x}{\cos} = \frac{2(c - x) \cos}{\cos 2}$$

Collecting x 's and c 's to opposite sides of the equation, we obtain

$$\begin{aligned} \frac{x}{2(c - x)} &= \frac{\cos^2}{\cos 2} \\ &= \frac{\cos^2}{2 \cos^2 - 1} \end{aligned}$$

As $\cos \neq 0$, $\frac{\cos^2}{2 \cos^2 - 1} \neq 1$, and so $x \neq \frac{2c}{3}$.