Parabola Volume Issue

UNS School Mathematics Competition
Junior Division Problems and Solutions

Solutions by **Denis Potapov**[1]

Problem

Every point on a line is painted using two different colours: black and white. Prove that there are always points **A**, **A** and **A** of the same colour such that

$$\mathbf{A}\mathbf{A} \quad \mathbf{A}\mathbf{A}.$$

o t on Choose any two points of the same colour, say black, **X** and **Y**. Let now **A** be the centre of **XY**; **B** be such that **X** is the centre of **BY** and **C** be such that **Y** is the centre of **CX**

Problem

Solve the equation

$$\sqrt{x} - \sqrt{x} \quad \sqrt{x-} - \sqrt{x-} .$$

o t on Write the equation in the form

$$\sqrt{x} \quad \sqrt{x-} \quad \sqrt{x-} \quad \sqrt{x} .$$

In such form, since

$$x \quad x- \quad x- \quad x \quad ,$$

after squaring of both sides, we arrive at

$$\sqrt{x} \times \sqrt{x-} \quad \sqrt{x-} \times \sqrt{x} .$$

Squaring again gives

$$x \quad \times \quad x- \quad x- \quad \times \quad x \quad \Leftrightarrow x - \quad x \quad .$$

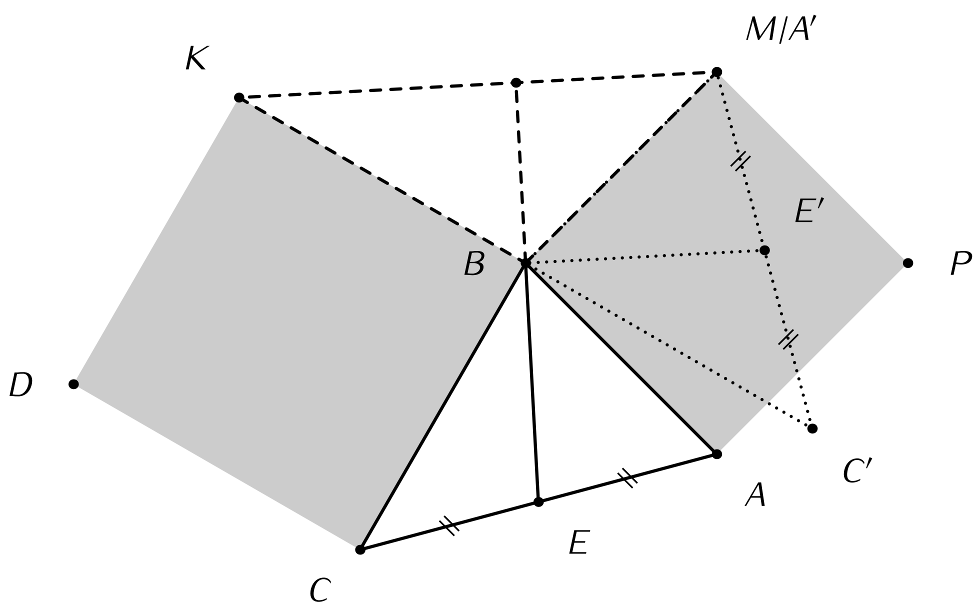
The latter solves to

$$x \quad \text{and } x \quad .$$

Problem 4

A triangle ABC has squares $ABMP$ and $BCDK$ built on its outer sides. Prove that the median BE of the triangle ABC is also an altitude of the triangle BMK.

o t on Rotate the triangle ABC by $^{\circ}$ around vertex B as shown on the picture below. After such transformation, the median BE becomes the mid-segment of the triangle KMC'. That is, on one hand, BE' is parallel to KM, and on the other hand, it is perpendicular to the median BE.



Problem
Find a five-digit number which equals times the product of its digits.

o t on Let

$$\mathbf{N} \quad \overline{abcde}$$

be the number. We are given that

$$\overline{abcde} \qquad a b c d e.$$

Note first that every digit a, b, c, d and e is odd. Indeed, otherwise, N

UNS School Mathematics Competition
Senior Division Problems and Solutions

Solutions by **Denis Potapov**[2]

Problem

A spherical planet has satellites. Prove that there is always a point on the surface of the planet such that at most satellites are seen from this point.

o t on Fix any two satellites, say S and S, and construct the plane through these satellites and the centre of the planet. Let A and B be the end points of the diameter of the planet perpendicular to this plane. The group of satellites visible from A does not intersect with the group of satellites visible from B. Moreover, the satellites S and S are also not visible from both point A and point B. Thus, at most

$$- \quad /$$

are visible from either point A or point B.

Problem

The sequence of numbers $\{a \}^{\infty}$ is such that

$$a$$

$$\text{and} \quad a \geq a \quad \frac{}{a}, \; k \quad , \; , \ldots$$

Prove that $a > $.

o t on Since $a - a \geq \frac{}{a}$, it follows that

$$a - a \quad a - a \times a \quad a \geq \frac{}{a} a \quad a \quad \frac{a}{a} \quad .$$

so

$$a > \sqrt{} > \; .$$

Problem

You are given a square table filled with positive integers. On every move, you are allowed to take from every element of a row; or to multiply every element of a column by . Prove that there is a strategy which can reduce every element in the table to zero.

o t on

$$a \quad b \quad c \qquad \Leftrightarrow a \quad b \quad c$$
$$a \quad b \quad c \qquad \Leftrightarrow a \quad b \quad c$$

However $a \quad b \quad c$ is odd, hence

$$a \quad b \quad c \qquad .$$

Finally,

$$\times \quad \times abc \leq$$

so

$$abc \leq \quad .$$

Thus, the choices for a, b and c are narrowed down to:

$$a, b, c$$

Let $\mathbf{CD}$ be the height of $\mathbf{ABC}$ and let $\mathbf{CE}$ be the diameter of the circle. Connect points $\mathbf{A}$ and $\mathbf{E}$. The triangle $\mathbf{CAE}$ is right-angled since the angle $\angle\mathbf{CAE}$ is rested on a diameter. Hence, the triangles

$$\mathbf{ACE} \text{ and } \mathbf{CDB}$$

are similar. The similarity implies that

$$\frac{\mathbf{CD}}{\mathbf{CB}} \quad \frac{\mathbf{AC}}{\mathbf{CE}}.$$

Solving for $\mathbf{CD}$ gives

$$\mathbf{CD} \quad \frac{\mathbf{CB} \times \mathbf{AC}}{\mathbf{CE}} \quad \frac{\times}{\times} \quad .$$

By Pythagoras' Theorem applied to $\mathbf{CDB}$,

$$\mathbf{DB} \quad \sqrt{\quad - \quad}$$

and, by By Pythagoras' Theorem on $\mathbf{ADC}$,

$$\mathbf{AD} \quad \sqrt{\quad - \quad} \quad .$$

Thus,

$$\mathbf{AB} \qquad .$$

Problem

Let $\mathbf{F}\ \mathbf{x}$ be a polynomial with integer coefficients and let

$$\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a}$$

be integers such that for any $\mathbf{n} \in \mathrm{N}$, there is an $\mathbf{a}_i$ such that $\mathbf{F}\ \mathbf{n}$ is a multiple of $\mathbf{a}_i$. Prove that there is one $\mathbf{a}_i$ such that $\mathbf{F}\ \mathbf{n}$ is a multiple of $\mathbf{a}_i$ for any $\mathbf{n} \in \mathrm{N}$.

ot The following theorem is known as **The Chinese Remainder Theorem** and it can be used in the solution of this problem without proof.

Theorem

Let

$$\mathbf{q}, \mathbf{q}, \ldots, \mathbf{q}_r \in \mathrm{Z}$$

be positive pairwise coprime integers. For any integers

$$\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}_r \in \mathrm{Z}$$

there is an integer $\mathbf{x} \in \mathrm{Z}$ such that

$$\mathbf{x} \equiv \mathbf{x}_i \quad \mathrm{od}\ \mathbf{q}_i,\ \mathbf{i} \quad, , \ldots, \mathbf{r}.$$