

**MATHEMATICS ENRICHMENT CLUB.**  
**Solution Sheet 18, October 1, 2018**

- Each tile has to cover one white and one black square, pairing up the squares into black-white couples. Removing opposite corners removes two squares of the same colour, so there are no longer equal numbers of black and white squares. Thus the tiling can't be done.
- Suppose that the three digit number has the form  $abc$  where  $a \in \{1; 2; \dots; 9\}$  and  $b; c \in \{0; 1; 2; \dots; 9\}$ . Then are trying to solve

$$100a + 10b + c = a! + b! + c! \quad (*)$$

Firstly, note that

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2 \\ 3! &= 6 \\ 4! &= 24 \\ 5! &= 120 \\ 6! &= 720 \\ 7! &= 5040 \end{aligned}$$

As  $7! = 5040$  is larger than 3 digits, none of 7, 8 or 9 may be used. Also,  $6! = 720$  and since our number cannot contain 7, 8 or 9, that means that 6 is also not possible. This gives an upper limit for the size of number of  $5! + 5! + 5! = 360$ . Also, note that  $4! + 4! + 4! = 72 < 100$ , so we need to have at least one 5.

Thus far, we know that  $a \in \{1; 2; 3\}$ , we must have at least one 5, and one other digit,  $x \in \{0; 1; 2; 3; 4; 5\}$ . If  $a = 1$ , for the RHS of (\*), we have

$$\begin{aligned} 1! + 5! + 0! &= 1 + 120 + 1 = 122 \\ 1! + 5! + 1! &= 1 + 120 + 1 = 122 \\ 1! + 5! + 2! &= 1 + 120 + 2 = 123 \\ 1! + 5! + 3! &= 1 + 120 + 6 = 127 \\ 1! + 5! + 4! &= 1 + 120 + 25 = 145 \\ 1! + 5! + 5! &= 1 + 120 + 120 = 241 \end{aligned}$$

We can see that the only option that works here is 145.

If  $a = 2$ , we have

$$2! + 5! + 0! = 2 + 120 + 1 = 123$$

$$2! + 5! + 1! = 2 + 120 + 1 = 123$$

$$2! + 5! + 2! = 2 + 120 + 2 = 124$$

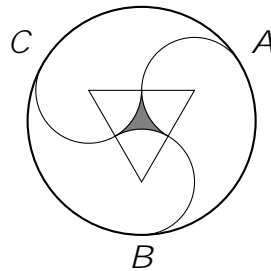
$$2! + 5! + 3! = 2 + 120 + 6 = 128$$

$$2! + 5! + 4! = 2 + 120 + 25 = 147$$

$$2! + 5! + 5! = 2 + 120 + 120 = 242$$

None of these work. That leaves us with  $a = 3$ , but in this case  $3! + 5! + 5! = 6 + 120 + 120 = 246 < 300$ . Hence the only number is 145.

3. This is closely related to Question (4) from last week. The simplest way to find the area of the tear-drop shape is to calculate the shaded area in the centre of the circle. Since the three tear-drops are congruent, if we subtract the shaded area from the area of the circle and then divide the result by three, we will have the area of each tear-drop.



We can connect the centres of the three arcs to form an equilateral triangle with side length  $2r$ . The area of this triangle is given by

$$\frac{1}{2}(2r)^2 \sin 60 = \sqrt{3}r^2$$

Each arc cuts out a sector of area  $\frac{1}{6} \pi r^2$  from the triangle. Thus the area of the shaded region is

$$\sqrt{3}r^2 - \frac{r^2}{2} = \frac{(2\sqrt{3} - 1)r^2}{2}$$

Recall that from last week that  $R = r(1 + \sqrt{3})$

If  $T$  is the area of one tear-drop, then

$$\begin{aligned}
 T &= \frac{1}{3} \frac{(7 + 4\sqrt{3})r^2}{3} - \frac{(2\sqrt{3})r^2}{2} \\
 &= \frac{r^2}{3} \frac{(7 + 4\sqrt{3})}{3} - \frac{(2\sqrt{3})}{2} \\
 &= \frac{r^2}{3} \frac{2(7 + 4\sqrt{3}) - 3(2\sqrt{3})}{6} \\
 &= \frac{r^2}{3} \frac{(17 + 8\sqrt{3})}{6}
 \end{aligned}$$

4. For a game to be a draw, all 9 squares must be filled without anyone winning along

## Senior Questions

1. (a) From the given definition,

$$\begin{aligned} \langle p_0; p_1 \rangle &= \int_{-1}^1 x dx \\ &= \frac{x^2}{2} \Big|_{-1}^1 \\ &= \frac{1}{2} - \frac{1}{2} = 0. \end{aligned}$$

(We can also use symmetry properties of an odd function to obtain this result.)

- (b) We want

$$\begin{aligned} \langle p_0; p_0 \rangle &= 1 \\ \int_{-1}^1 (x^0)^2 dx &= 1 \\ \int_{-1}^1 1 dx &= 1 \\ 2(x^0)^2 &= 1 \\ x^0 &= \frac{1}{\sqrt{2}}. \end{aligned}$$

And

$$\begin{aligned} \langle p_1; p_1 \rangle &= 1 \\ \int_{-1}^1 (x^1)^2 dx &= 1 \\ \int_{-1}^1 x^2 dx &= 1 \\ \frac{2}{3} (x^1)^2 &= 1 \\ x^1 &= \frac{\sqrt{3}}{2}. \end{aligned}$$

- (c) Suppose that  $q_2 = a_2x^2 + a_1x + a_0$ , where  $a_0$ ,  $a_1$  and  $a_2$  are to be determined. If  $\langle q_2; q_0 \rangle = 0$ , then

$$\begin{aligned} \int_{-1}^1 (a_2x^2 + a_1x + a_0) dx &= 0 \\ \int_{-1}^1 (a_2x^2 + a_1x + a_0) dx &= 0: \end{aligned}$$

By symmetry,  $\int_{-1}^1 a_1x dx = 0$  for any value of  $a_1$ .

Similarly,

$$\begin{aligned} \int_1^2 (a_2 x^2 + a_0) dx &= 2 \int_0^1 (a_2 x^2 + a_0) dx \\ &= 2 \left[ a_2 x^3 + a_0 x \right]_0^1 \end{aligned}$$