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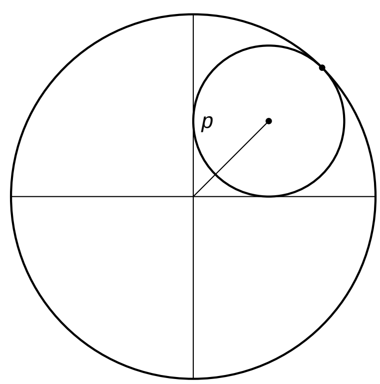
MATHEMATICS ENRICHMENT CLUB.

Problem Sheet 17 Solutions, September 30, 2019

1. If a number is divisible by 8, then its last three digits are a multiple of 8. This gives us $b = 4$. If a number is divisible by 9, then the sum of its digits is also divisible by 9. This gives us $a = 9$. Thus $a + b = 9 + 4 = 13$.

2. Suppose that the smaller circle has radius r. Then the distance between the centres of the two circles is $\sqrt{2}r$. Thus the radius of the big circle is $(1+\sqrt{2})r$.



Senior Questions

1. Rearranging the given equation,

$$\begin{aligned} 3x^2 - 8y^2 + 3x^2y^2 &= 2008 \\ 3x^2(1+y^2) &= 8(251+y^2) \\ 3x^2 &= \frac{8(250+1+y^2)}{1+y^2} \\ 3x^2 &= 8\left(1+\frac{250}{1+y^2}\right) \end{aligned}$$

Since the RHS is an integer, this means that $1 + y^2$ is a factor of 250. The factors of 250 are 1, 2, 5, 10, 25, 50, 125, 250, which gives possible values of y of 1, 2, 3, and 7. However, the only one of these values that gives a multiple of 3 is $y = 7$, which gives $x = 4$.

2. Since f is a polynomial, we can write it as

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n = \sum_{k=0}^{n} a_kx^k,$$

where $a_0, \dots, a_n$ are non-negative integers. Furthermore,

$$f(1) = a_0 + a_1 + a_2 + \dots + a_n = \sum_{k=0}^{n} a_k.$$

Since $f(1) = 6$, this tells us that at most 6 of the a_k are non-zero, and also that no coefficient is larger than 6. This means that we can find the coefficients of f by writing 3438 in base 7, which is the same as writing 3438 in terms of integer multiples of powers of 7.

Using the change of base algorithm we discussed in the solutions to Problem Sheet 16,

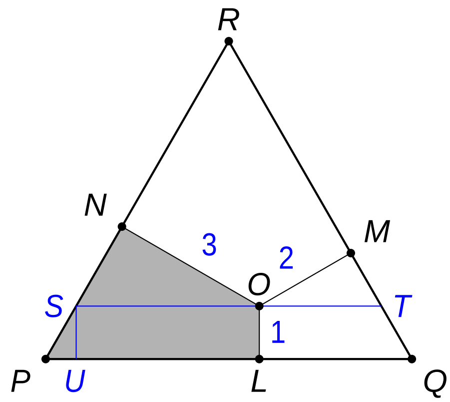
$$\begin{aligned} 3438 &= 491 \times 7 + 1 \\ 491 &= 70 \times 7 + 1 \\ 70 &= 10 \times 7 + 0 \\ 10 &= 1 \times 7 + 3 \\ 1 &= 0 \times 7 + 1. \end{aligned}$$

So

$$\begin{aligned} 3438 &= (13011)_7 \\ \Longrightarrow \quad f(7) &= 1 \times 7^4 + 3 \times 7^3 + 0 \times 7^2 + 1 \times 7^1 + 1 \times 7^0 \\ \Rightarrow \quad f(x) &= x^4 + 3x^3 + x + 1. \end{aligned}$$

Evaluating f at $x = 3$, we obtain 166.

3. We rescale the triangle so that $OL = 1$, $OM = 2$, and $ON = 3$.



Firstly, we will find the area of the shaded region. Draw the line ST, parallel to PQ and passing through O, and let SU be the perpendicular from S to PQ. Then $\triangle RST$ is also an equilateral triangle, and using some basic trigonometry we can calculate that $SO = 2\sqrt{3}$, $SN = \sqrt{3}$, $PU = 1/\sqrt{3}$.

Then $SOLU$ has area $2\sqrt{3}$; $\triangle SPU$ has area $\frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$; and $\triangle NSO$ has area $\frac{3\sqrt{3}}{2}$. Thus

$$\text{Area of } LONP = 2\sqrt{3} + \frac{\sqrt{3}}{6} + \frac{3\sqrt{3}}{2} = \frac{11\sqrt{3}}{3}.$$

We will now calculate the area of $\triangle PQR$. Recall from Question 5 on Problem Sheet 10, 2018, that the sum of the perpendiculars from any interior point of an equilateral triangle is equal to the altitude of the triangle. Thus we know that $\triangle PQR$ has altitude 6, side length $4\sqrt{3}$, and hence area $12\sqrt{3}$.

Thus

$$\frac{a}{b} = \frac{11\sqrt{3}/3}{12\sqrt{3}} = \frac{11}{36},$$

and so $a +$