MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 9, July 8, 2019

1. Given a n-digit long number, if we x the last digit (say let it be 1), then there are $(n \quad 1)!$ ways to arrange the other $n \quad 1$ digits (say $2; 3; : : : ; n$) to get a di erent n-digit long number. Hence, each of $1; 2; : : : ; n$ will appear Td [(2)]TJ/F19 11.9552 Tf9

4.

Senior Questions

1. (a) Firstly, we want to make the RHS of (1) look like the RHS of (2). Thus

$$j\frac{d^2}{dt^2} + c\frac{d}{dt} = I_{motor}$$
$$Rj\frac{d^2}{dt^2} + (Rc + k_m)\frac{d}{dt} = R \quad I_{motor} + k_m\frac{d}{dt}$$

Hence

$$Rj\frac{d^2}{dt^2} + (Rc + k_m)\frac{d}{dt} = V_{in}. \qquad (3)$$

Let $! = \frac{d}{dt}$. As $! \quad ! \quad !_1$, $\frac{d^2}{dt^2} \quad ! \ 0$. So

$$\begin{aligned} !_1 &= \frac{V_{in}}{Rc + k_m} \\ &= \frac{12}{10(1) + 5} \\ &= 0.8 \text{ rads/s} \end{aligned}$$

(b) Similarly, if we set $!(0) = 0$, then

$$\begin{aligned} \frac{d^2}{dt^2} &= \frac{V_{in}}{Rj} \\ &= \frac{12}{(10)(5)} \\ &= 0.24 \text{ rads/s}^2 \end{aligned}$$

(c) If we wish to solve (3) for (t), we rst re-write in terms of $!$. Then

$$\begin{aligned} Rj\frac{d!}{dt} + (Rc + k_m)! &= V_{in} \\ \frac{d!}{dt} + \frac{Rc + k_m}{Rj}! &= \frac{V_{in}}{Rj} \end{aligned}$$

is called the *method of integrating factors*.) Consequently,

$$\begin{aligned}
e^{t(Rc+k_m)/Rj}\frac{d\omega}{dt} + \frac{Rc+k_m}{Rj}e^{t(Rc+k_m)/Rj}\omega &= \frac{V_{in}}{Rj}e^{t(Rc+k_m)/Rj} \\
\frac{d}{dt}\left(\omega e^{t(Rc+k_m)/Rj}\right) &= \frac{V_{in}}{Rj}e^{t(Rc+k_m)/Rj} \\
\omega e^{t(Rc+k_m)/Rj} &= \frac{V_{in}}{Rj}\int e^{t(Rc+k_m)/Rj}\,dt \\
&= \frac{V_{in}}{Rj}\left(\frac{Rj}{Rc+k_m}e^{t(Rc+k_m)/Rj}\right) + C_1 \\
&= \frac{V_{in}}{Rc+k_m}e^{t(Rc+k_m)/Rj} + C_1 \\
\omega &= \frac{V_{in}}{Rc+k_m} + C_1e^{-t(Rc+k_m)/Rj}
\end{aligned}$$

Using the initial condition given in (b), we have $C_1 = -\frac{V_{in}}{Rc+k_m}$, and hence

$$\omega(t) = \frac{V_{in}(1-e^{-t(Rc+k_m)/Rj})}{Rc+k_m}.$$

Since $\omega = \frac{d\theta}{dt}$,

$$\begin{aligned}
\theta(t) &= \frac{V_{in}}{Rc+k_m}\int (1-e^{-t(Rc+k_m)/Rj})\,dt \\
&= \quad V_{in}\, k
\end{aligned}$$

In particular, consider the angle bisector, BD. Let $\backslash EDB =$. Then $\backslash EDB =$ $\backslash BDF =$. Furthermore, since $\backslash BDF$ and $\backslash FEB$ subtend the same arc, $\backslash FEB =$. Similarly, it can be shown that $\backslash BFE =$, which implies that $4BEF$ is isosceles with $EB = BF$. As OB and OF are both radii of the circle, $4OEF$ is also isoceles. This means that $EOFB$ is a kite with diagonals EF and OB that intersect perpendicularly at M. Now since O is the centre of the circumcircle, OM is the perpendicular bisector of EF.